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SOURCE ENCODING IN THE PRESENCE OF RANDOM DISTURBANCE

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Let us consider the source encoding situation analyzed by Shannon [1, 2]. The random quantities involved are shown in Figure 1. The quantity $\underset{\sim}{M}$ denotes a vector-valued random variable, which represents the output of an ergodic source generated in an interval of time of T seconds duration. The vector-valued random variable $\underset{\sim}{M}$ takes on values in some Hilbert space $\mathcal{M}$; specific vectors in this space are denoted by $\underset{\sim}{m}$. Similarly, the receiver output corresponding to $\underset{\sim}{M}$ is denoted by $\underset{\sim}{\hat{\hat{Z}}}$; $\underset{\sim}{\hat{\hat{Z}}}$ takes on values in a Hilbert space $\mathcal{Z}$ and specific vectors in $\mathcal{Z}$ are denoted by $\underset{\sim}{z}$. The quantities $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ denote respectively the channel input and output corresponding to $\underset{\sim}{M}$. Our interest here is in the encoding of the source output, hence we assume that either the channel is disturbance free so that $\underset{\sim}{Y} = \underset{\sim}{X}$ or that encoding is used on the channel so that $\underset{\sim}{Y} = \underset{\sim}{X}$ with probability $1 - \delta$, $\delta \ll 1$. The spaces $\mathcal{M}$ and $\mathcal{Z}$ may, for example, be the spaces $L_2\,[0, T]$ or Euclidean m-space E_m in which m is proportional to T.

Let us now focus our attention on the nature of the ergodic source. In previous work it is usually implicitly implied that $\underset{\sim}{M}$ represents an uncorrupted version of some message of direct interest to the user. In this instance we denote the quantity $\underset{\sim}{M}$ by $\underset{\sim}{Z}$, as shown in Figure 2, to indicate the direct relationship between the nature of the receiver output and the source output. The distortion is then the expected value of $\rho\,(\underset{\sim}{z}, \underset{\sim}{\hat{\hat{z}}})$, an appropriately chosen distance function of $\underset{\sim}{z}$ and $\underset{\sim}{\hat{\hat{z}}}$.

$$D\,(\underset{\sim}{M}, \underset{\sim}{Z}) = E\,\rho\,(\underset{\sim}{Z}, \underset{\sim}{\hat{\hat{Z}}}) \tag{1}$$

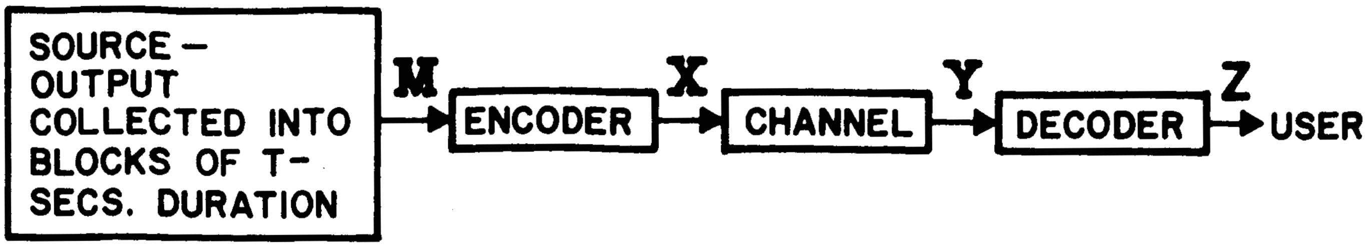

FIGURE 1
SOURCE ENCODING SITUATION

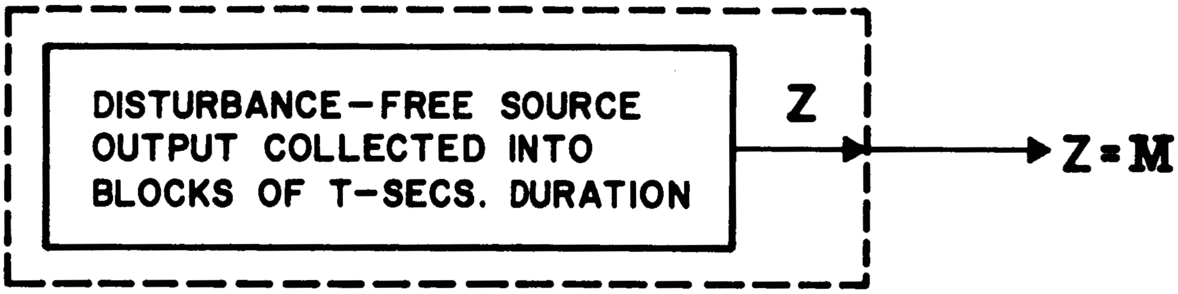

FIGURE 2
DISTURBANCE-FREE SOURCE

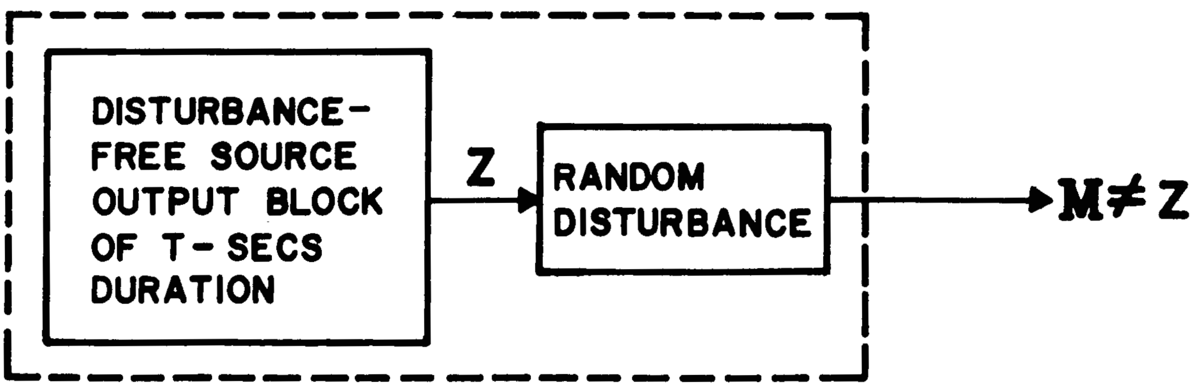

FIGURE 3
SOURCE WITH RANDOM DISTURBANCE

In space experimentation, a more common situation is the one shown in Figure 3; the observable source output is not the quantity of direct interest, $\underset{\sim}{Z}$, but is instead some randomly corrupted version of $\underset{\sim}{Z}$, which we denote by $\underset{\sim}{M}$. In this case, our distortion measure is again of the form given by equation (1) since $\underset{\sim}{Z}$ is the quanitity of direct interest to the experimenter or user at the receiver.

The question to which we address ourselves is the following: what is the relationship between the situations of Figures 2 and 3? In particular, if one has considered the disturbance free source of Figure 2 and devised a good code, can this knowledge be applied to yield a good code for the noisy or corrupted source of Figure 3?

A partial answer to this question is as follows. In the particular case of mean square error

$$D(\underset{\sim}{M}, \hat{\hat{\underset{\sim}{Z}}}) = E\left\{ \| \underset{\sim}{Z} - \hat{\hat{\underset{\sim}{Z}}} \|^2 \right\} \tag{2}$$

the problem of source encoding in the presence of random disturbance can be resolved into solution of

1) Estimation of $\underset{\sim}{Z}$ from $\underset{\sim}{M}$
2) Encoding of $\hat{\underset{\sim}{Z}}$, the optimum estimate of $\underset{\sim}{Z}$ based on observation of $\underset{\sim}{M}$, as if it were a disturbance free source output.

To prove this assertion, we start by considering the structure of the optimum code when the encoder operates directly on $\underset{\sim}{M}$. Consider a set of encoding vectors $\underset{\sim}{z}_1, \underset{\sim}{z}_2, \ldots, \underset{\sim}{z}_n$ in the space $\mathcal{Z}$ and the corresponding encoding regions for the distortion measure of equation 2

$$\mathcal{M}_k = \left\{ \underset{\sim}{m} : E\left\{ \| \underset{\sim}{Z} - \underset{\sim}{z}_k \|^2 | \underset{\sim}{M} = \underset{\sim}{m} \right\} \leq E\left\{ \| \underset{\sim}{Z} - \underset{\sim}{z}_j \|^2 \mid \underset{\sim}{M} = \underset{\sim}{m} \right\} \text{all } j \neq k \right\} \tag{3}$$

with points equidistant from two or more $\underset{\sim}{z}_k$ being arbitrarily assigned to one of the corresponding sets. An encoder then simply notes which set $\mathcal{M}_k$ an output vector $\underset{\sim}{m}$ falls into and then transmits the index of this set (either directly over an error free channel or with coding over a noisy channel). Let H_n denote the entropy of the discrete random variable denoting the index of the set. The distortion using this transmission system is then

$$D = E \left\{ \| \underset{\sim}{Z} - \hat{\underset{\sim}{Z}} \|^2 \right\} = E \left\{ \| \underset{\sim}{Z} - \underset{\sim}{z}_k \|^2 \right\} \tag{4}$$

in which k is a random variable denoting for a given randomly generated $\underset{\sim}{Z}$ and $\underset{\sim}{M}$ the index of the set $\mathcal{M}_k$ corresponding to $\underset{\sim}{M}$. Shannon's theory of source encoding (see [2], especially pp. 109-111 and 116-120) states the following. Given a value D^* ($\geq D_{min}$) and a value $\epsilon > 0$, then for a sufficiently large block time, T, it is possible to choose the number n and the vectors $\underset{\sim}{z}_1, \underset{\sim}{z}_2, \ldots, \underset{\sim}{z}_n$ such that

1) D as given by equation (3) has the value D^*
2) The entropy of the discrete random variable denoting the index k is less than or equal to $T\,(R(D^*) + \epsilon)$, in which $R(D)$ is the rate distortion function.

Further, the negative portion of Shannon's source encoding theorem (see [2], pp. 102- 05) states that no method of transmission using capacity $C = R(D^*)$ can improve on the performance of this system. Such an encoding procedure is visualized in Figure 4.

Now let us show that this encoding procedure may be split into the two steps shown in Figure 5. Let $\hat{\underset{\sim}{Z}} = \hat{\underset{\sim}{z}}(\underset{\sim}{M})$ denote the minimum mean-square-error estimate of $\underset{\sim}{Z}$ given $\underset{\sim}{M}$ ($\hat{\underset{\sim}{Z}}$ denotes the estimate, $\hat{\underset{\sim}{z}}(\ )$ the estimator or function determining the optimum estimate).

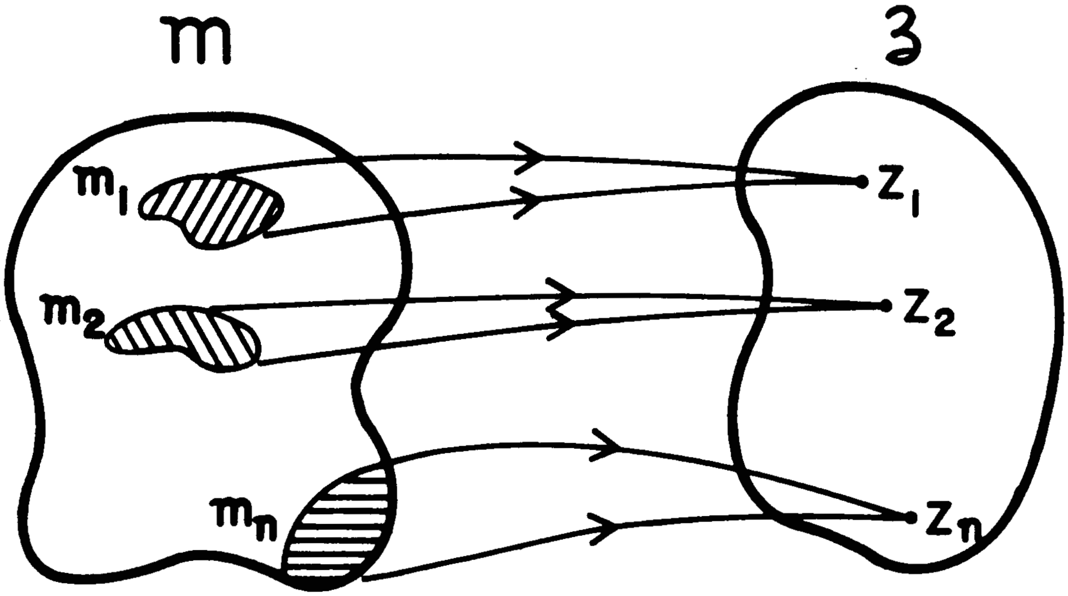

FIGURE 4
VISUALIZATION OF ENCODING MAPPING

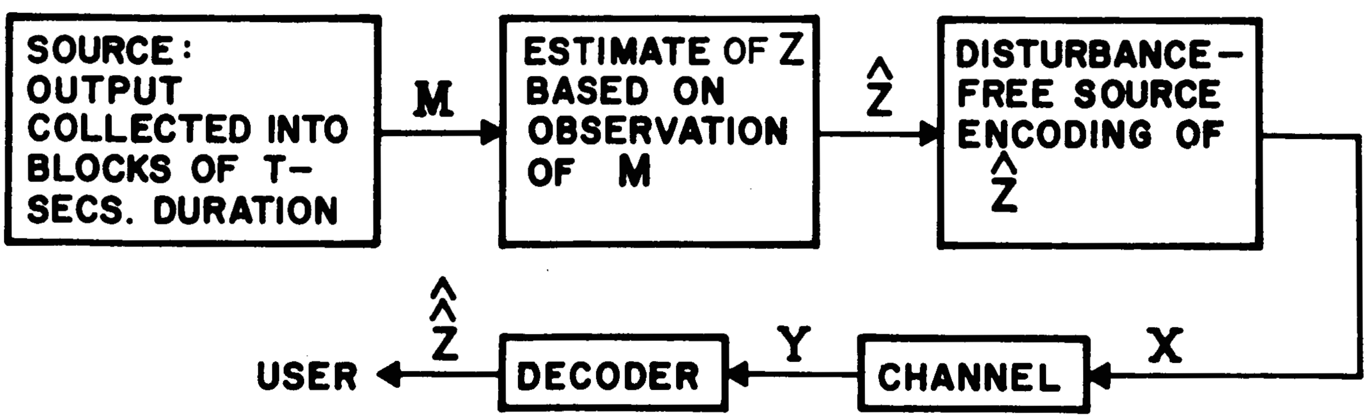

FIGURE 5
SOURCE ENCODING SPLIT INTO:
1) ESTIMATION OF Z
2) DISTURBANCE-FREE ENCODING OF $\hat{Z}$

Now the optimum such estimate is determined by the fact that the error $\underset{\sim}{Z} - \hat{\underset{\sim}{Z}}$ is uncorrelated with <u>any</u> function of $\underset{\sim}{M}$. Thus for any $\underset{\sim}{z}_q$

$$
\begin{aligned}
&E\left\{\| \underset{\sim}{Z} - \underset{\sim}{z}_q \|^2 \mid \underset{\sim}{M} = \underset{\sim}{m}\right\} \\
&\quad = E\left\{\| \underset{\sim}{Z} - \hat{\underset{\sim}{Z}} + \hat{\underset{\sim}{Z}} - \underset{\sim}{z}_q \|^2 \mid \underset{\sim}{M} = \underset{\sim}{m}\right\} \\
&\quad = E\left\{\| \underset{\sim}{Z} - \hat{\underset{\sim}{z}}(\underset{\sim}{m}) \|^2 + 2(\underset{\sim}{Z} - \hat{\underset{\sim}{z}}(\underset{\sim}{m}), \hat{\underset{\sim}{z}}(\underset{\sim}{m}) - \underset{\sim}{z}_q) \right. \\
&\qquad\qquad \left. + \| \hat{\underset{\sim}{z}}(\underset{\sim}{m}) - \underset{\sim}{z}_q \|^2 \mid \underset{\sim}{M} = \underset{\sim}{m}\right\} \\
&\quad = E\left\{\| \underset{\sim}{Z} - \hat{\underset{\sim}{z}}(\underset{\sim}{m}) \|^2 \mid \underset{\sim}{M} = \underset{\sim}{m}\right\} + \| \hat{\underset{\sim}{z}}(\underset{\sim}{m}) - \underset{\sim}{z}_q \|^2 \qquad (5)
\end{aligned}
$$

Using equation (5) in both sides of inequality (3), we see that the encoding region for the k^{th} coding vector, z_k, is determined by

$$\mathcal{M}_k = \{m: \ \| \hat{\underset{\sim}{z}}(m) - \underset{\sim}{z}_k \|^2 \leq \| \hat{\underset{\sim}{z}}(m) - \underset{\sim}{z}_j \|^2, \text{ all } j \neq k \qquad (6)$$

But this statement says that the index to be transmitted can be determined by

1) <u>estimating $\underset{\sim}{Z}$ from the observation $\underset{\sim}{M} = \underset{\sim}{m}$</u>
2) <u>picking the index of the coding vector closest to $\hat{\underset{\sim}{z}}(\underset{\sim}{m})$.</u>

From equation (5) it also follows immediately that the overall distortion is given by

$$D = E\left\{\| \underset{\sim}{Z} - \underset{\sim}{z}_k \|^2\right\} = E\left\{\| \underset{\sim}{Z} - \underset{\sim}{z}(\underset{\sim}{M}) \|^2\right\} + E\left\{\| \underset{\sim}{z}(\underset{\sim}{M}) - \underset{\sim}{z}_k \|^2\right\} \quad (7)$$

which is just the sum of the estimation error at the transmitter (which is independent of the choice of code vectors) plus the error involved in source encoding the estimate as if it were disturbance free. <u>Thus the set of code vectors that minimizes D is the same set that minimizes the distortion due to encoding $\hat{\underset{\sim}{Z}}$ as if it were disturbance free.</u>

Thus the optimum code when the message is observed in the presence of random disturbance can be designed by first finding the optimum estimate of the quantity of direct interest to the user and then source encoding this estimate as if it were disturbance free. In conclusion, it should be pointed out that this simplification is a direct consequence of the choice of mean-square-error and does not apply in general. If, for example, one takes D as given by equation (1) with

$$\rho(\underset{\sim}{z}, \hat{\hat{\underset{\sim}{z}}}) = \begin{cases} 1 & \| \underset{\sim}{z} - \hat{\hat{\underset{\sim}{z}}} \| > \varepsilon_0 \\ 0 & \| \underset{\sim}{z} - \hat{\hat{\underset{\sim}{z}}} \| \leq \varepsilon_0 \end{cases} \tag{8}$$

then the above result no longer holds; the encoding must be done directly on the $\mathcal{M}$ space with the nature of the disturbance being taken into account in determining the encoding vectors and regions.

ACKNOWLEDGEMENTS

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[1] C.E. Shannon, "A Mathematical Theory of Communication," Bell System Tech. J., Vol. 17, pp. 623-656, Oct. 1948. See especially part V.

[2] C.E. Shannon, "Coding Theorems for a Discrete Source with a Fidelity Criterion," in Information and Decision Processes, Robt. E. Machol, Ed., McGraw Hill Book Co., Inc., New York, 1960.